Int. J. Heat Mass Transfer. Vol. 34, No. 3, pp. 875–885, 1991
Printed in Great Britain

0017-9310/91 \$3.00+0.00

Natural convection heat transfer from plates of finite dimensions

WITOLD M. LEWANDOWSKI
Department of Chemical Engineering, Technical University of Gdańsk, ul. Majakowskiego 11/12, 80-952 Gdańsk, Poland

(*Received* 2 *May* 1989 *and in final form* 27 *December* 1989)

Abstract—A new approach to the model of natural convection from an isothermal inclined plate and a simplified analytical solution of this model are presented. In this model two separate regions with different fluid motions are distinguished. In the first region, the direction of fluid flow inside the boundary layer and parallel to the plate buoyancy force component converge, while in the second one these directions are opposite. The theory presented is verified experimentally.

1. INTRODUCTION

A CONSIDERABLE discrepancy between the results of theoretical considerations and experimental investigations occurs in the convection heat transfer from flat isothermal surfaces. This discrepancy is not constant, but it alters with the plate inclination angle. Differences between particular criteria relations, describing the vertical case, obtained by 25 authors, differ by $\pm 20\%$ [1, 2]. For inclined plates, the discrepancies are much greater and amount to $\pm 45\%$ within the laminar range and to about $\pm 100\%$ within the turbulent range [3]. The interval containing the results obtained for the horizontal case by 19 investigators amounts to *c.* $\pm 50\%$ [1–3].

In the hitherto investigations [4] it has been proved that the methods of measurement and the equipment employed may cause errors yet their magnitude is constant and independent of the plate inclination angle. The only exception is the gradient method, in which the accuracy may be influenced by the surface inclination angle, especially in the case of incompetent or accidental location of the experimental points (thermocouples) on the measurement surface [5]. This method, however, is most frequently used as a qualitative method together with a quantitative (balance) method, so it cannot be the main reason for the discrepancy of the above-mentioned criteria relations. Therefore, it has been decided to search for the reasons of such a behaviour by verifying the correctness of a physical model of the phenomenon. In the author's opinion even the most accurate calculations based on an inaccurate model are not so valuable as approximate calculations based on the contrary on a more accurate model.

2. AIM OF WORK

The presented research attempts to apply a model of the convective heat transfer, which would be universal for all angles of plate inclinations $(0 < \phi < \pi/2)$. The proposed model is a consequence of visualization of the convective heat transfer from real inclined plates. The aim was to describe the phenomena taking place in the presented model by reduced differential equations and also to solve them in a simplified way. Approximate calculation methods used in the works of an experimental type are adequate due to the comparable order of accuracy of the obtained results. Moreover, the advantage of these methods is the possibility of direct interpretation of the obtained results and a quick modification of experimental investigations.

3. THEORETICAL MODELS OF THE CONVECTION HEAT TRANSFER

The model proposed by Schmidt and Beckmann [6] belongs to the already classical models of convective heat transfer. This model was obtained on a basis of experimental results of visualization of the boundary thermal layer on a vertical isothermal plate [7]. Figure 1(a) presents a graphical interpretation of this model. Subsequent investigations on vertical plates were carried out by Lorenz (1934), Saunders (1939), Schuh (1948), Ostrach (1953), Sparrow (1959), Gebhart (1962, 1966) and also by Fujii (1972) [9], Takeuchi (1974) [10], Ling (1982) [11], Churchill (1983) [12] and others [8]. They are characterized by increasing accuracy of calculations due to elimination of consecutive simplifying assumptions or to defining them more accurately due to addition of other limiting conditions. These considerations, however, irrespective of the fact whether they have been conducted analytically or numerically, are always based on the same physical model of this phenomenon (Fig. 1(a)).

The same model (Fig. 1(b)) has also been adopted for a description of the results obtained with inclined plates. The boundary layer thickness increases along the plate length, but the buoyancy force, present in the Navier–Stokes equations, is replaced by the force

NOMENCLATURE

a	thermal diffusivity, $\lambda/c_p\rho$
c_p	specific heat at constant pressure of fluid
d	length or diameter of the plate
F	coefficient of boundary layer shape, equation (12)
g	gravitational acceleration
H	height of a column of fluid (see Fig. 5)
i	specific enthalpy
m	mass flux
$Nu_{(d)}$, $Nu_{(x)}$	Nusselt numbers, $\alpha d/\lambda$, $\alpha x/\lambda$, respectively
p	pressure
q	heat flux density
Q	heat flux
$Ra_{(d)}$, $Ra_{(x)}$, $Ra_{(\delta)}$	Rayleigh numbers, $g\beta\Delta Td^3/\nu a$, $g\beta\Delta Tx^3/\nu a$, $g\beta\Delta T\delta^3/\nu a$, respectively
T	temperature
W	velocity
x	length of boundary layer
x	coordinate horizontal to the surface
y	coordinate vertical to the surface.

Greek symbols

α	heat transfer coefficient
β	coefficient of volumetric expansion
δ	thickness of boundary layer
Δ	difference
θ	dimensionless temperature
λ	thermal conductivity
ν	kinematic viscosity
ρ	fluid density
ϕ	angle of plate inclination
Φ	coefficient related to ϕ and F, equation (29).

Subscripts

ch	characteristic
cr	point of separation of boundary layers
I, II	region I or region II
tot	total
w	wall
x	x-direction or to fluid properties inside boundary layer
∞	fluid ambient condition.

components x and y. This case has been investigated by Rich (1953), Sugawa (1955), Vilet (1969) [8], Kierkus (1968) [13] and also by Hassan (1970) [14], Fujii (1972) [9], Miller (1978) [15], Rasmus (1979) [16], Raithby (1983) [12], Lewandowski (1986, 1987) [3] and others.

Convective heat transfer from a horizontal plate has also been explained by a similar model (Fig. 1(c)) based on a homogeneous boundary layer increasing on a semi-infinite flat surface. The work of Rotem and Claassen (1969) [17] is a standard example of utilization of this model for a horizontal plate. These scientists have also published the results of visualization investigations. Their photographs not only show the initial edge of the horizontal plate, they also show the place where the boundary layer transforms into a plume. At this point, the physical model of this phenomenon accepted by them starts to be inadequate. Hence, according to the analysis of their results of visualization and also of Schmidt's earlier photographs (1934) [18] the analogy between the growth of a boundary thermal layer on a vertical and a horizontal plate does not concern the whole surface. Other investigators of this case, based on a semi-infinite plate, are for example: Pera (1973), Blanc (1974) [8], Goldstein (1983) [19] and others.

The results of visualization experiments, carried out on real plates by Al-Arabi (1976) [20], Sparrow (1969),

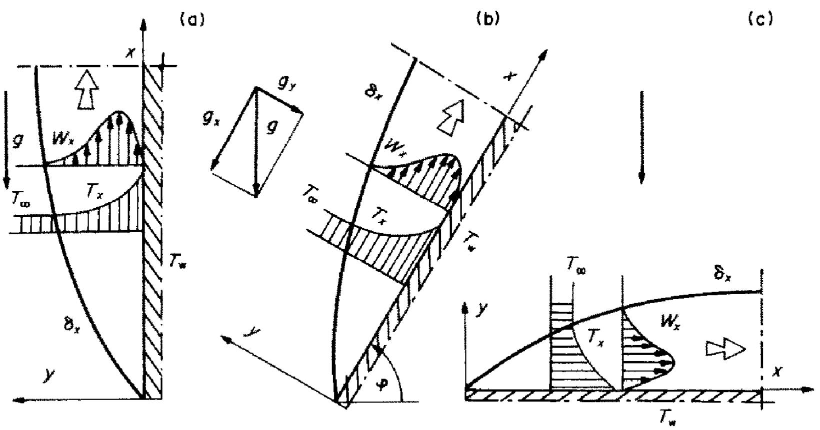

FIG. 1. Theoretic models of natural convection for a flat surface.

Keun-Shik (1988) [21] and others [3] suggest that it is necessary to employ another model of the phenomenon.

However, to our knowledge, no works devoted to horizontal surfaces, based on any other model than that of the semi-infinite plate, have been published.

4. RESULTS OF VISUALIZATION OF NATURAL CONVECTION FROM INCLINED PLATES

The results of visual research on round plates presented in this chapter do not refer to the entire range of plate inclination angles, but they are confined to small values of these angles, satisfying the condition of axial symmetry. At greater angles ($\phi > 12$ deg) the experiments have been carried out on rectangular plates, which for this range approached more closely the two-dimensional model of the phenomenon [5]. Photographs presented in Fig. 2 concern the convective heat transfer from an isothermal flat round plate of diameter $d = 0.07$ m. From among the photographs of plate inclination angle ranging from $\phi = 0$ to 12 deg the case of plate inclination angle $\phi = 4$ deg has been chosen as an example for presentation. The tested fluid was glycerine and heating fluxes were $q = 2.514$ (Fig. 2(a)) and 7.288 kW m^{-2} (Fig. 2(b)).

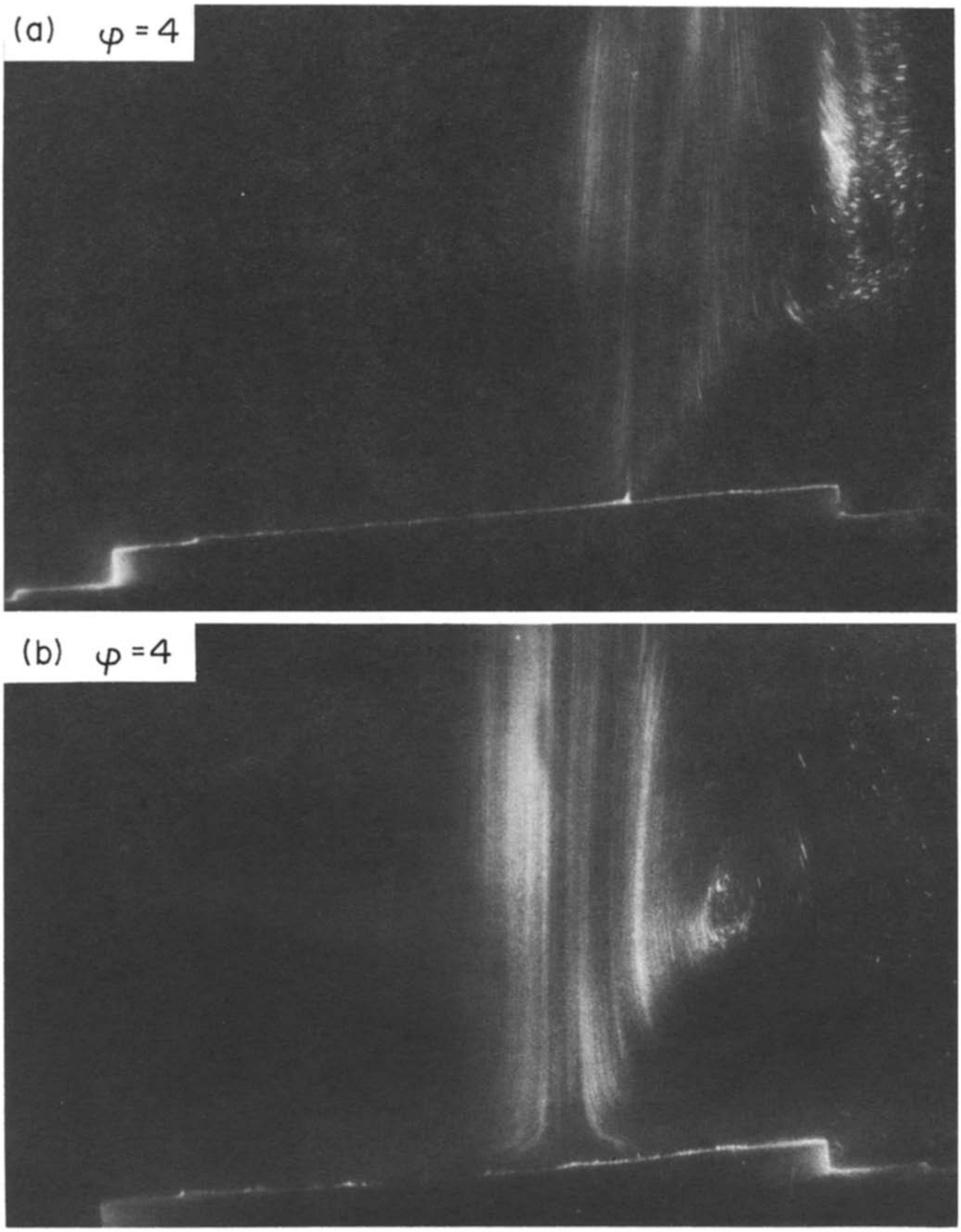

FIG. 2. Visual photographs of natural convection heat transfer from an isothermal, round ($d = 0.07$ m) plate to glycerine. Exposure time $\tau = 5$ s: (a) $q = 2.514$ kW m^{-2}; (b) $q = 7.288$ kW m^{-2}.

Table 1. Results of experimental study of natural convection heat transfer from an isothermal, round and inclined plate to glycerine

ϕ [deg]	T_w [°C]	T_∞ [°C]	α [W m^{-2} deg^{-1}]	$Nu_{(d)}$ [—]	$Ra_{(d)}$ [—]
			$q = 2.514$ kW m^{-2}		
0	32.57	15.67	89.993	22.072	6.206E+6
1	32.57	15.91	91.264	22.383	6.206E+6
2	32.59	16.09	92.176	22.605	6.224E+6
3	32.47	16.28	93.917	23.032	6.134E+6
4	32.57	16.49	94.605	23.199	6.202E+6
5	32.80	16.75	94.744	23.231	6.379E+6
6	32.94	16.97	95.162	23.333	6.486E+6
8	33.11	17.25	95.868	23.504	6.612E+6
10	33.60	17.46	94.191	23.090	7.017E+6
12	33.81	17.60	93.781	22.988	7.199E+6
			$q = 7.288$ kW m^{-2}		
0	54.28	20.45	130.246	31.797	5.478E+7
1	54.17	20.35	130.337	31.820	5.418E+7
2	54.07	20.17	129.975	31.733	5.359E+7
3	54.07	19.98	129.257	31.559	5.339E+7
4	54.21	19.79	128.021	31.257	5.378E+7
5	54.07	19.51	127.498	31.132	5.289E+7
6	54.07	19.25	126.550	30.902	5.260E+7
8	53.86	18.85	125.870	30.739	5.129E+7
10	53.30	18.57	126.893	30.993	4.874E+7
12	53.20	18.21	125.954	30.767	4.799E+7

The quantitative aspect of the obtained experimental results has been presented in Table 1.

The above photographs and the experimental procedure of investigation on natural convection heat transfer from a round, inclined plate, as well as the equipment used in the research, have been more extensively described in refs. [3–5].

Analysis of the photographs of all the cases of plate inclination angle gives evidence that for a horizontal plate the boundary layers grow identically from opposite leading edges and then transform above the plate into a plume. The centreline of this axially symmetric free heat flux is vertical to the surface and passes through the plate symmetry axis. At increased inclination angle the separation point of the boundary layers, through which passes the centreline, moves to one (trailing) edge and the opposite boundary layers (identical for the horizontal case) begin to differ increasingly from each other.

5. PROPOSED MODELS OF NATURAL CONVECTION FROM FLAT FINITE PLATES

In the suggested models (Fig. 3) of convective heat transfer from flat isothermal surfaces of finite dimensions, transition from one case to the other proceeds together with a displacement of the separation point. Thus, the case of an inclined plate is virtually a general model concerning plates arbitrarily oriented in an unlimited space (Fig. 3(b)) in which vertical (Fig. 3(a)) and horizontal (Fig. 3(c)) positions of plates constitute only specific cases. Versatility of the suggested model consists of the fact that a change in surface inclination angle results in fluent changes of the participation of regions I and II in the heat exchange, whereas participation of region III remains constant. For the two characteristic cases—horizontal and vertical—the differences between regions I and II gradually disappear in the first case, whereas in the second case it is region II that gradually disappears. In the final effect for horizontal plates two symmetrical boundary layers exist, growing from each edge (region I = region II) and transforming at the separation point into free stream heat convection (plume). On the other hand, in the case of vertical plates the model is identical to the hitherto applied one, because then only one boundary layer appears (region II = 0) subsequently converting into a wake and next into a plume (region III) [22].

6. PHYSICAL MODEL OF THE PHENOMENON

Limiting the considerations only to the boundary layer region and to the two-dimensional case a physical model of the phenomenon may be expressed as follows (Fig. 4). In this model three regions of the convective heat transfer are specified.

(1) Region I, in which the buoyancy force is parallel to g_x and its sense conforms to fluid velocity W_x. It is a region of decisive importance with regard to heat transfer, because the thickness of a boundary layer (δ_x) is smaller, whereas its length (x) is greater than in region II.

(2) Region II differs from region I in the sense of fluid velocity in the boundary layer and in consequences resulting from this fact as, for instance, different shapes of the boundary layer $\partial\delta/\partial x$ and x.

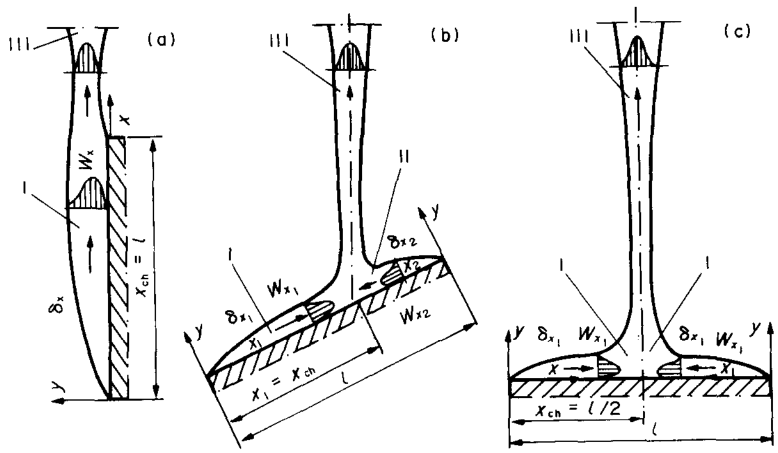

FIG. 3. Real models of the convection heat transfer from: (a) vertical; (b) inclined; (c) horizontal plates.

(3) Region III, or the free heat convective flux, which according to the evidence reported in refs. [5, 22], as it departs from the plate becomes the same for all cases and therefore this region was omitted in the calculations presented below.

7. SIMPLIFIED QUASI-ANALYTICAL SOLUTION

Introducing the simplifying assumptions typical for the natural convection such as:

(a) fluid is incompressible and its flow is laminar,

(b) inertia forces in comparison with the viscosity forces may be ignored,

(c) $W_x \gg W_y$,

(d) physical properties of fluid in boundary layers (index x) and in the undisturbed region (∞) are constant,

(e) the temperature of the plate (T_w) is constant,

(f) thicknesses of thermal and hydraulic boundary layers are the same, the Navier–Stokes equations may be written as

$$\nu \frac{\partial^2 W_x}{\partial y^2} \pm g\beta(T_x - T_\infty)\sin\phi - \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad (1)$$

$$\beta(T_x - T_\infty)g\cos\phi - \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \qquad (2)$$

where the + and − signs refer to region I and region II, respectively.

Instead of the direct form of the Fourier–Kirchhoff equation it was decided, according to Squir and Eckert, to make the assumption that the temperature profile in the boundary layer is described by

$$\theta = (T_x - T_\infty)/(T_w - T_\infty) = (1 - y/\delta)^2. \qquad (3)$$

In this quasi-analytical method the continuity equation was only used to estimate the correctness of the obtained results.

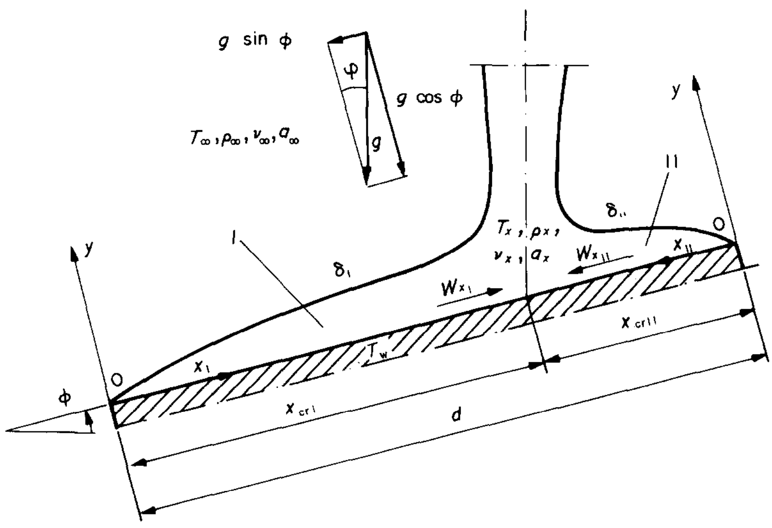

FIG. 4. Two-dimensional model of a physical phenomenon.

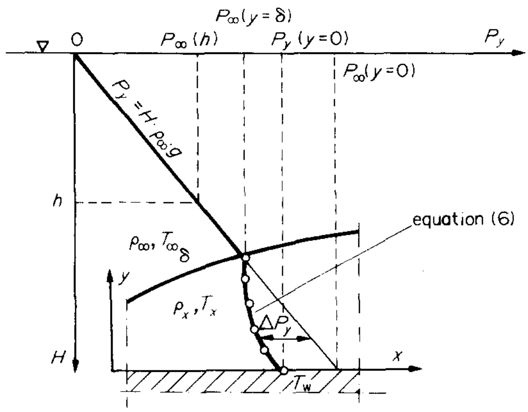

FIG. 5. Graphic interpretation of equation (6).

A substitution of equation (3) into equations (1) and (2) gives

$$\nu\frac{\partial^2 W_x}{\partial y^2}\pm\beta\Delta Tg\left(1-\frac{y}{\delta}\right)^2\sin\phi-\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \quad (4)$$

$$\beta\Delta Tg\left(1-\frac{y}{\delta}\right)^2\cos\phi-\frac{1}{\rho}\frac{\partial p}{\partial y}=0. \quad (5)$$

Integration of equation (5) for the boundary condition ($y=\delta$, $p_y=p_{x(y=\delta)}$) gives a formula for the pressure distribution in a boundary layer directed vertical to the heating surface

$$p_y=p_{x(y=\delta)}+\rho\beta\Delta Tg\left(y-\frac{y^2}{\delta}+\frac{y^3}{3\delta^2}-\frac{\delta}{3}\right)\cos\phi. \quad (6)$$

Figure 5 presents the physical interpretation of equation (6) on an example of a horizontal plate.

Pressure ($p_{x(y=\delta)}$), described by equation (7) represents the excess of pressure over the hydrostatic pressure ($p'_{x(y=\delta)}$), existing on the border of the boundary layer on the following level:

$$p'_{x(y=\delta)}=(H-\delta\cos\phi-x\sin\phi)\rho_\infty g. \quad (7)$$

Comparison of both these pressures is possible after taking into account thermal expansion of fluid

$$\rho=\rho_\infty[1-\beta(T_x-T_\infty)] \quad (8)$$

which yields

$$p_{x(y=\delta)}=p'_{x(y=\delta)}=(H-\delta\cos\phi-x\sin\phi)\rho_\infty g$$
$$=\rho g\beta(T_x-T_\infty)(H-\delta\cos\phi-x\sin\phi). \quad (9)$$

In a case of natural convection in an unlimited space, when $H\gg\delta$ and $H\gg x$, equation (9) transforms into

$$p_{x(y=\delta)}=\rho g\beta(T_x-T_\infty)H\cong\text{const.} \quad (10)$$

From condition (10), equation (6) may be differentiated with respect to x

$$\frac{\partial p}{\partial x}=\rho g\beta\Delta T\left(\frac{y^2}{\delta^2}-\frac{2y^3}{3\delta^3}-\frac{1}{3}\right)\frac{\partial\delta}{\partial x}\cos\phi. \quad (11)$$

For subsequent considerations in pursuance of refs. [9, 13] the mean value of the boundary layer thickness increase on the length of its growth was introduced and it was assumed that this value, except the separation point ($x=x_{cr}$) and above the leading edge of the plate ($x=0$), is constant

$$\frac{\partial\delta}{\partial x}\cong\frac{\partial\delta}{\partial x}=F\cong\text{const.} \quad (12)$$

Substitution of equations (11) and (12) into equation (4) leads to

$$\nu\frac{\partial^2 W_x}{\partial y^2}=g\beta\Delta TF_{(\mathrm{I}/\mathrm{II})}\left(\frac{y^2}{\delta^2}-\frac{2y^3}{3\delta^3}-\frac{1}{3}\right)\cos\phi$$
$$\pm g\beta\Delta T\left(1-\frac{y}{\delta}\right)^2\sin\phi. \quad (13)$$

For the boundary condition (for $y=0$ and δ, $W_x=0$) a double integration of equation (13) allows the evaluation of the formula of local (equation (14)) and mean (equation (15)) velocity in the boundary layer

$$W_{x(\mathrm{I}/\mathrm{II})}=\frac{g\beta\Delta T}{\nu}\left[F_{(\mathrm{I}/\mathrm{II})}\left(\frac{y^4}{12\delta^2}-\frac{2y^5}{60\delta^3}-\frac{y^2}{6}+\frac{7\delta y}{60}\right)\right.$$
$$\left.\times\cos\phi\pm\left(-\frac{y^2}{2}+\frac{y^3}{3\delta}-\frac{y^4}{12\delta^2}+\frac{\delta y}{4}\right)\sin\phi\right] \quad (14)$$

$$\bar{W}_{x(\mathrm{I}/\mathrm{II})}=\frac{1}{\delta}\int_0^\delta W_{x(\mathrm{I}/\mathrm{II})}\,\mathrm{d}y$$
$$=\frac{g\beta\Delta T\delta^2_{(\mathrm{I}/\mathrm{II})}}{\nu}\left[F_{(\mathrm{I}/\mathrm{II})}\frac{\cos\phi}{72}\pm\frac{\sin\phi}{40}\right]. \quad (15)$$

A change in mass flow intensity caused by a change in fluid density gives

$$\mathrm{d}m=\mathrm{d}(\rho\bar{W}_x 1\delta)=\frac{3\rho g\beta\Delta T}{\nu}\left[F_{(\mathrm{I}/\mathrm{II})}\frac{\cos\phi}{72}\pm\frac{\sin\phi}{40}\right]$$
$$\times\delta^2_{(\mathrm{I}/\mathrm{II})}\,\mathrm{d}\delta. \quad (16)$$

The amount of heat necessary to create this change is

$$\mathrm{d}Q=\Delta i\,\mathrm{d}m=\frac{3\rho g\beta\Delta T}{\nu}\left[F_{(\mathrm{I}/\mathrm{II})}\frac{\cos\phi}{72}\pm\frac{\sin\phi}{40}\right]$$
$$\times\delta^2_{(\mathrm{I}/\mathrm{II})}(\overline{T_x-T_\infty})c_p\,\mathrm{d}\delta. \quad (17)$$

Substitution of the mean value of temperature

$$(\overline{T_x-T_\infty})=\frac{1}{\delta}\int_0^\delta\Delta T\left(1-\frac{y}{\delta}\right)^2\mathrm{d}y=\frac{\Delta T}{3} \quad (18)$$

yields

$$\mathrm{d}Q=\frac{\rho g\beta(\Delta T)^2}{\nu}\left[F_{(\mathrm{I}/\mathrm{II})}\frac{\cos\phi}{72}\pm\frac{\sin\phi}{40}\right]\delta^2_{(\mathrm{I}/\mathrm{II})}c_p\,\mathrm{d}\delta. \quad (19)$$

The heat flux may be subordinated to the heat transfer coefficient (α) availing of equation (20)

$$dQ = \alpha 1 \Delta T \, dx = -\lambda \left(\frac{\partial \theta}{\partial y}\right)_{y=0} 1 \Delta T \, dx. \quad (20)$$

From the adopted temperature profile inside the boundary layer (equation (3)) the dimensionless temperature gradient on the wall may be evaluated as

$$\left(\frac{\partial \theta}{\partial y}\right)_{y=0} = -\frac{2}{\delta}. \quad (21)$$

Substituting equation (21) into equation (20) and equating the result with equation (19), the dependence (22) and equation (23) are obtained

$$\frac{\rho g \beta \Delta T}{2\lambda v}\left[F_{(I/II)}\frac{\cos\phi}{72} \pm \frac{\sin\phi}{40}\right]\delta^3_{(I/II)} c_p \, d\delta = dx. \quad (22)$$

The thickness of the boundary layer may be determined from integration of equation (22) for the boundary condition ($x = 0, \delta = 0$)

$$\delta_{(I/II)} = \frac{8^{1/4} x_{(I/II)}}{(Ra_{(x_{I/II})})^{1/4}\left[F_{(I/II)}\frac{\cos\phi}{72} \pm \frac{\sin\phi}{40}\right]^{1/4}}. \quad (23)$$

The other way of finding the thickness of the boundary layer by substituting correlation (12) into equation (22) has also been considered, yet the roots of the obtained quadratic equation of F according to the criteria given by Czebyszew have no exact solutions.

From equations (20) to (23) the local and subsequently mean value of heat transfer coefficient (α) in region I (subscript I and +) or in region II (II and −) are obtained

$$\alpha_{(I/II)} = \frac{2^{1/4}\lambda}{x_{(I/II)}}[Ra_{(x_{I/II})}]^{1/4}\left[F_{(I/II)}\frac{\cos\phi}{72} \pm \frac{\sin\phi}{40}\right]^{1/4} \quad (24)$$

$$\bar{\alpha}_{(I/II)} = \frac{1}{x_{cr(I/II)}}\int_0^{x_{cr(I/II)}} \alpha_{(I/II)} \, dx = \frac{2^{9/4}\lambda}{3x_{(I/II)}}[Ra_{(x_{I/II})}]^{1/4}\left[F_{(I/II)}\frac{\cos\phi}{72} \pm \frac{\sin\phi}{40}\right]^{1/4}. \quad (25)$$

The complete heat flux transferred from the plate is

$$Q_{tot} = Q_I + Q_{II} \quad (26)$$

which may also be written as

$$\overline{\alpha_{tot}} \Delta T(x_I + x_{II}) = \bar{\alpha}_I \Delta T x_I + \bar{\alpha}_{II} \Delta T x_{II}. \quad (27)$$

A subsequent transformation yields

$$\frac{\overline{\alpha_{tot}}(x_I + x_{II})}{\lambda} = \frac{2^{9/4}}{3}\left(\frac{g\beta\Delta T}{va}\right)^{1/4}(x_I^{3/4}\Phi_I^{1/4} + x_{II}^{3/4}\Phi_{II}^{1/4}) \quad (28)$$

where

$$\Phi_{(I/II)} = F_{(I/II)}\frac{\cos\phi}{72} \pm \frac{\sin\phi}{40}. \quad (29)$$

Assuming that in the place of separation of the boundary layers and their transformation into a free convection flux the thicknesses of these layers in regions I and II are equal ($\delta_{crI} = \delta_{crII}$), correlation (30) is obtained from equation (23)

$$\frac{x_I}{x_{II}} = \frac{\Phi_I}{\Phi_{II}} = \frac{F_{II}}{F_I}. \quad (30)$$

By substitution of dependence (30) in equation (28), equation (31) is obtained as criteria relation (32) pertinent for any inclination of the plate, despite the fact that it is described exclusively with parameters of region I

$$\frac{\overline{\alpha_{tot}} x_I}{\lambda} = \frac{2^{9/4}}{3}(Ra_{(x_I)})^{1/4}\Phi_I^{1/4} \quad (31)$$

$$Nu_{(x_I)} = 1.586(Ra_{(x_I)}\Phi_I)^{1/4}. \quad (32)$$

The solved criteria equation (32) may be utilized after determination of the relation for the region I boundary layer length (x_I) as a characteristic linear dimension of a phenomenon and of the relation for the value of coefficient F_I in region I.

For a plate of a length or diameter d the following dependence is correct:

$$x_I + x_{II} = d. \quad (33)$$

Use of it in equation (30) yields:

$$x_I = d\left(1 - \frac{1}{2 + \dfrac{g\tan\phi}{5F_I}}\right). \quad (34)$$

The value of coefficient F_I may, however, be determined according to its definition (12) from equation (23)

$$F_I = \frac{1}{x_I}\int_0^{x_I}\frac{d\delta}{dx}dx = \frac{1}{x_I}\delta\Big|_0^{x_I} = \frac{2^{3/4}}{(Ra_{(x_I)}\Phi_I)^{1/4}}. \quad (35)$$

Substitution of equation (35) into equation (34) gives

$$x_I = d\left[1 - \frac{1}{2 + 3^2 \cdot 2^{-3/4} \cdot 5^{-1} \cdot \tan\phi\,[Ra_{(x_I)}\Phi_I]^{1/4}}\right]. \quad (36)$$

The above obtained solution seems to be inconsistent with the hitherto known criteria relations for horizontal, vertical and inclined plates and it additionally gives an impression that it is too complicated and thus difficult for practical use. In subsequent sections these doubts will be explained.

8. PRACTICAL UTILIZATION OF THE ANALYTICAL SOLUTION

The obtained result of the simplified analytical solution resolves to the system of four mutually correlated equations with four unknowns (37)–(40)

$$F = \frac{2^{3/4}}{(Ra_{(ch)}\Phi)^{1/4}} \tag{37}$$

$$\Phi = F\frac{\cos\phi}{72} + \frac{\sin\phi}{40} \tag{38}$$

$$Ra_{(ch)} = \frac{g\beta\Delta T x_{(ch)}^3}{\nu a} \tag{39}$$

$$x_{(ch)} = d\left[1 - \frac{1}{2+3^2\cdot 2^{-3/4}\cdot 5^{-1}\cdot \tan\phi\,(Ra_{(ch)}\Phi)^{1/4}}\right]. \tag{40}$$

The simultaneous solution of these equations had been numerically carried out based on an algorithm shown in Fig. 6. For the introduced values of Rayleigh number ($Ra_{(d)}$) and inclination angle (ϕ), the values F, Φ, $x_{(ch)}$ and $Ra_{(ch)}$ were obtained.

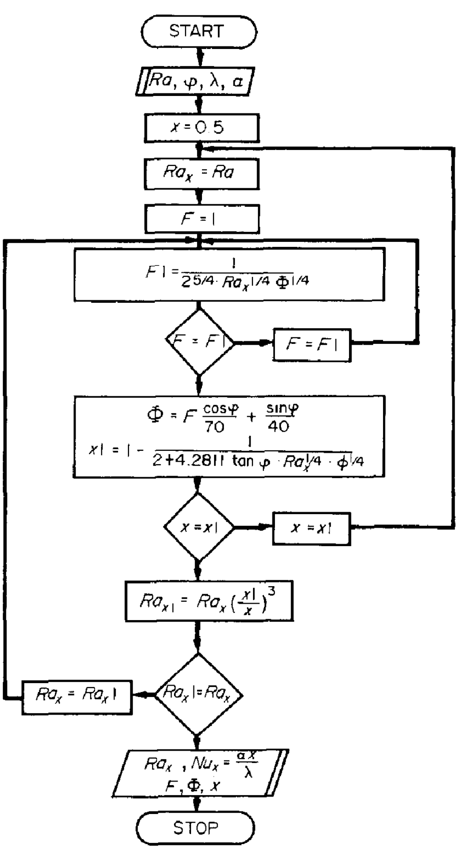

FIG. 6. Algorithm of calculation of equations (37)–(40).

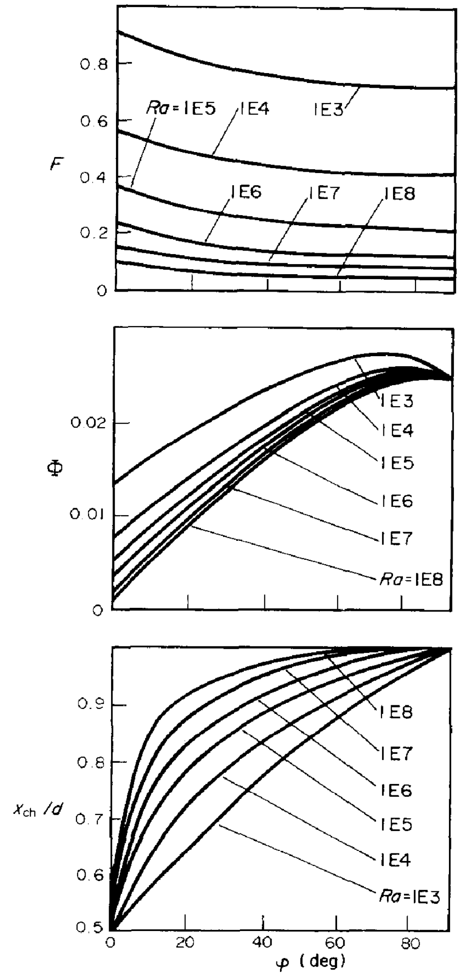

FIG. 7. Dependence of the angle (ϕ) and the Rayleigh number (Ra) on: the coefficient F, coefficient Φ and the characteristic linear dimension x_{ch}.

The results of calculations made in this way are presented in a graphic form in Fig. 7.

In the calculation of our own experimental investigations the Nusselt number should also be determined based on the appointed characteristic linear dimension ($x_{(ch)}$). The quoted algorithm renders possible reckoning of other investigators' results, elaborated in the form applied so far. It is necessary then to know additionally the dimensions of the heating plates used by them (d) to determine $Nu_{(ch)}$ according to the relation

$$Nu_{(ch)} = Nu_{(d)}\frac{x_{(ch)}}{d}. \tag{41}$$

9. EXPERIMENTAL VERIFICATION

Experimental investigations on natural convection from a rectangular plate (0.1 m, 0.06 m) published by the author in 1984 [4] led to the following criteria relations:

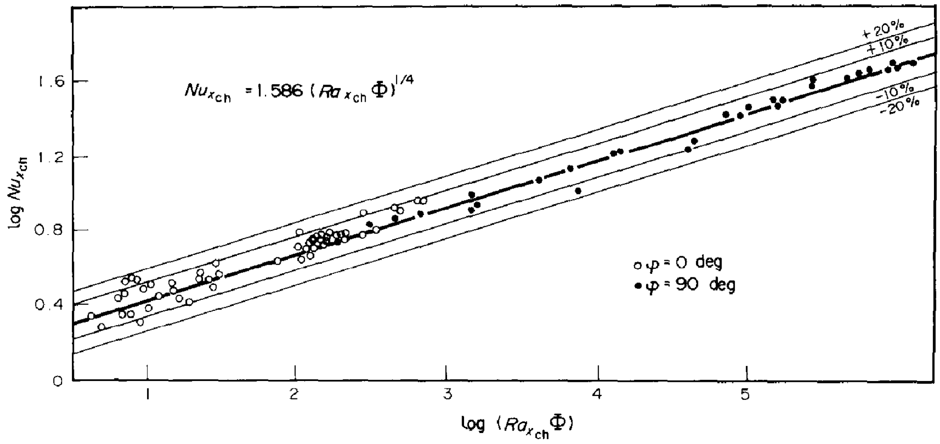

FIG. 8. Results of experimental investigations [4] elaborated in accordance with the theory suggested in this paper.

$$Nu = 0.612(Ra)^{1/4} \quad \text{for a vertical plate, } \phi = 90 \text{ deg} \tag{42}$$

$$Nu = 0.766(Ra)^{1/5}$$

$$\text{for a horizontal plate, } \phi = 0 \text{ deg.} \tag{43}$$

Re-elaboration of the obtained results according to a procedure presented in Section 8 is shown in Fig. 8.

The effect of utilization of the suggested theory for recalculating the results obtained for a round inclined plate (Table 1) is presented in Table 2 and Fig. 9.

The results of experimental investigations on an inclined plate (ϕ = 45, 60, 75 and 90 deg) carried out by Kierkus [23] had also been elaborated in a similar way. The results of these experiments, dealing with air, had been presented in a form of the relations of local Nusselt and Grashof numbers, therefore the Rayleigh numbers have been determined initially and subsequently they have been recalculated in accordance with the given algorithm. The effect is presented in Fig. 10.

Analysis of Figs. 8–10 yields evidence that about 95% of the experimental results falls within a ±20% range, hence it may be stated that verification of the suggested theory yielded positive results.

10. CONCLUSIONS

The suggested model of the phenomenon of convective heat transfer from real, isothermal, inclined

Table 2. Experimental results from Table 1 recalculated using the theory presented in this paper

ϕ [deg]	x_{ch}/d [—]	F [—]	Φ [—]	$Nu_{(x_{ch})}$ [—]	$Ra_{(x_{ch})}$ [—]
		$q = 2.514$ kW m^{-2}			
0	0.50000	0.15614	0.00217	11.04	7.757E+5
1	0.54729	0.15033	0.00252	12.25	1.017E+6
2	0.58893	0.14524	0.00289	13.31	1.271E+6
3	0.62509	0.14133	0.00327	14.40	1.498E+6
4	0.65723	0.13716	0.00364	15.25	1.761E+6
5	0.68598	0.13292	0.00402	15.94	2.059E+6
6	0.71113	0.12939	0.00440	16.59	2.332E+6
8	0.75282	0.12363	0.00518	17.69	2.821E+6
10	0.78711	0.11764	0.00595	18.17	3.423E+6
12	0.81396	0.11333	0.00674	18.71	3.882E+6
		$q = 7.288$ kW m^{-2}			
0	0.50000	0.10101	0.00140	15.90	6.847E+6
1	0.57052	0.09564	0.00176	18.15	1.006E+7
2	0.62793	0.09138	0.00214	19.93	1.327E+7
3	0.67475	0.08776	0.00253	21.29	1.640E+7
4	0.71337	0.08452	0.00291	22.30	1.952E+7
5	0.74462	0.08218	0.00332	23.18	2.184E+7
6	0.77090	0.07997	0.00372	23.82	2.410E+7
8	0.81139	0.07659	0.00453	24.94	2.740E+7
10	0.84039	0.07439	0.00536	26.05	2.893E+7
12	0.86314	0.07208	0.00618	26.56	3.086E+7

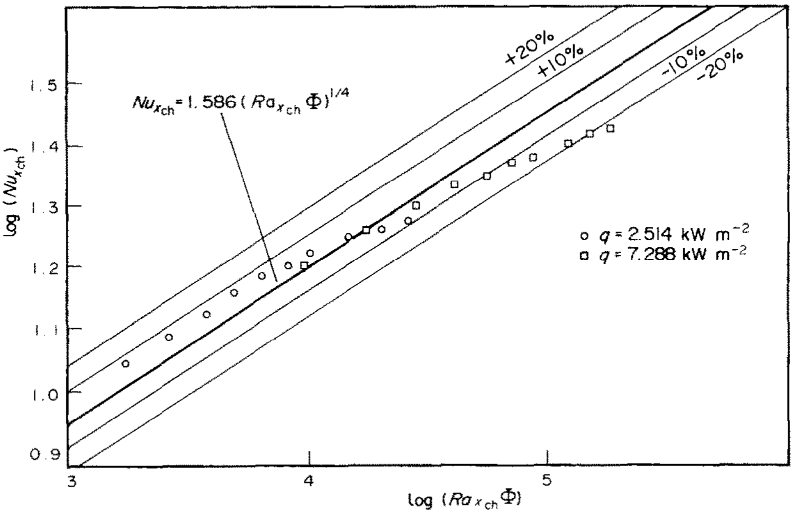

FIG. 9. Graphical form of experimental results presented in Table 2.

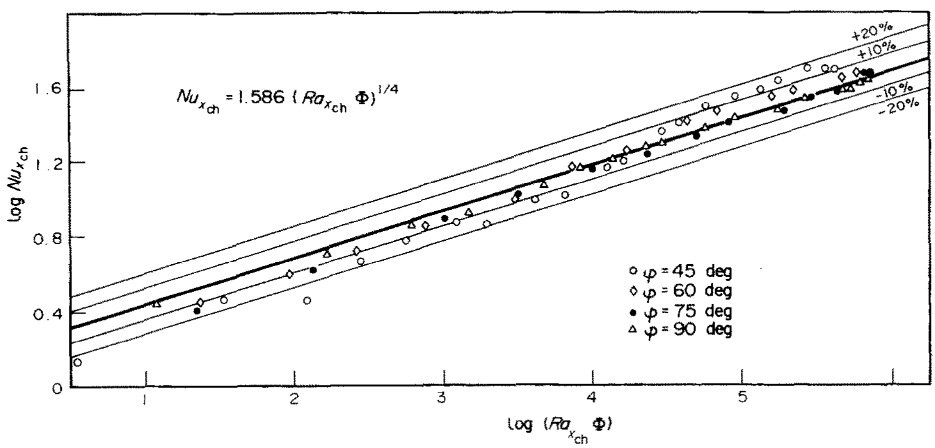

FIG. 10. Results of experimental investigations [23] elaborated in accordance with the suggested theory.

plates exhibits a convergence with the results of experimental investigations which cannot be only accidental. Even the initial results suggest that it is possible to describe all the cases of heating plate inclination with one criteria relation and then to compare the results on a single graph. Evidently, both the two-dimensional model, as well as its simplified solution, are merely a first approximation of the real phenomenon and this simplified solution may very likely inspire other investigators to solve more accurately the simultaneous partial differential equations describing the above presented model with methods proposed by Kierkus [13], Chen *et al.* [24], Yu and Lin [25] or one of the numerical methods, for instance that of Takeuchi *et al.* [10].

Acknowledgement—I would like to express my deep gratitude to Professor Jarosław Mikielewicz for his help and inspiring discussions concerning the presented theory.

REFERENCES

1. W. M. Lewandowski and P. Kubski, Methodical investigation of free convection from vertical and horizontal plates, *Wärme- und Stoffübertr.* **17**, 147 (1983).
2. W. M. Lewandowski, Systematyczne badania przejmowania ciepła od płyty poziomej do cieczy w przestrzeni nieograniczonej w warunkach konwekcji naturalnej (Systematic investigations of heat transfer from a horizontal plate to a fluid in an unlimited space in the conditions of natural convection), Doctor's Thesis, Technical University of Gdańsk (1980).
3. W. M. Lewandowski, Wymiana ciepła od płaskich powierzchni w warunkach laminarnej konweckji swobodnej (Heat transfer from flat surfaces under conditions of laminar free convection), *Zesz. Nauk. Politech. Gdańsk., t. Mechanika-53* **412,** 1–90 (1987).
4. W. M. Lewandowski and P. Kubski, Effect of the use of the balance and gradient methods as a result of experimental investigation of natural convection action with regard to the conception and construction of measuring apparatus, *Wärme- und Stoffübertr.* **18,** 247 (1984).
5. W. M. Lewandowski, The mechanism of natural con-

vection heat transfer from isothermal vertical, horizontal and inclined plates, *Arch. Termod* 7(4), 235 (1986).
6. E. Schmidt und W. Beckmann, Das Temperatur- und Geschwindigkeitsfeld von einer wärmeabgebenden senkrechten Platte bei natürlicher Konvektion, *Techn. Mech. Thermodyn.* 341–391 (1930).
7. E. Schmidt, *V.D.I., Forschung* **3**, 181 (1932).
8. Y. Jaluria, *Natural Convection Heat and Mass Transfer*. Pergamon Press, Oxford (1980).
9. T. Fujii and H. Imura, Natural convection heat transfer from a plate with arbitrary inclination, *Int. J. Heat Mass Transfer* **15**, 755 (1972).
10. M. Takeuchi, Y. Ota and Y. Tanaka, Laminar natural convection heat transfer from a flat plate in a thermally stratified fluid, *Trans. Jap. Soc. Mech. Engrs* **40**, 1046 (1974).
11. F. F. Ling, A simple laboratory apparatus for measurements of natural convection heat transfer over a flat plate in vertical, inclined and horizontal positions, *Int. J. Mech. Engng Educ.* **11**(4), 245 (1982).
12. S. W. Churchill, Free convection around immersed bodies. In *Singlephase Convective Heat Transfer*. Hemisphere, Washington, DC (1983).
13. W. Kierkus, An analysis of laminar free convection flow and heat transfer about an inclined isothermal plate, *Int. J. Heat Mass Transfer* **11**, 241 (1968).
14. K.-E. Hassan and S. A. Mohamed, Natural convection from isothermal flat surfaces, *Int. J. Heat Mass Transfer* **13**, 1873 (1970).
15. R. M. Miller and B. Gebhart, An experimental study of the natural convection flow over a heated ridge in air, *Int. J. Heat Mass Transfer* **21**, 1220 (1978).
16. A. Rasmus, Wymiana ciepła od płyty dowolnie zorientowanej w przestrzeni nieograniczonej w warunkach konwekcji naturalnej dla czynników o dużej liczbie Prandtla (Heat transfer from a plate of any orientation in an unlimited space under conditions of natural convection for great Prandtl number factors), Doctor's Thesis, Technical University of Gdańsk (1979).
17. Z. Rotem and L. Claassen, Natural convection above unconfined horizontal surfaces, *J. Fluid Mech.* **38**(1), 173–192 (1969).
18. E. Schmidt, *Proc. IVth Int. Congr. for Applied Mech.*, Cambridge, p. 92 (1934).
19. R. J. Goldstein and K.-S. Lau, Laminar natural convection from a horizontal plate and the influence of plate edge extensions, *J. Fluid Mech.* **129**, 55 (1983).
20. M. Al-Arabi and M. El-Riedy, Natural convection heat transfer from isothermal horizontal plates of different shapes, *Int. J. Heat Mass Transfer* **19**, 1399 (1976).
21. K.-S. Chang and J.-C. Han, Natural convective heat transfer over inclined short plates, *Commun. Appl. Numer. Meth.* **4**, 665–673 (1988).
22. W. M. Lewandowski, Konwekcyjne i hydrodynamiczne strumienie swobodne (Jets and plumes), *Inż. Chem. i Proc.* **6**(4), 695 (1985).
23. W. Kierkus, Analiza konwekcji swobodnej dookoła skośnej izotermicznej powierzchni płaskiej (Analysis of free convection around an inclined, isothermal flat surface), Doctor's Thesis, Technical University of Warsaw (1965).
24. T. S. Chen, H. C. Tien and B. F. Armaly, Natural convection on horizontal and vertical plates with variable surface temperature or heat flux, *Int. J. Heat Mass Transfer* **29**, 1465–1478 (1986).
25. W.-S. Yu and H.-T. Lin, Free convection heat transfer from an isothermal plate with arbitrary inclination, *Wärme- und Stoffübertr.* **23**, 203–211 (1988).

CONVECTION THERMIQUE NATURELLE POUR DES PLAQUES DE DIMENSION FINIE

Résumé—On présente une nouvelle approche par un modèle de convection naturelle pour une plaque isotherme inclinée et une solution analytique simplifiée est donnée. On distingue deux régions séparées avec des mouvements différents de fluide. Dans la première région, l'écoulement dans la couche limite et la composante de la force de flottement parallèle à la plaque convergent tandis que la seconde zone ces directions sont opposées. La théorie présentée est vérifiée expérimentalement.

WÄRMEÜBERGANG DURCH NATÜRLICHE KONVEKTION AN ENDLICHEN PLATTEN

Zusammenfassung—Es wird ein neues Modell für die natürliche Konvektion an einer isothermen geneigten Platte sowie eine vereinfachte analytische Lösung dafür vorgestellt. In diesem Modell werden zwei getrennte Gebiete mit unterschiedlicher Fluidbewegung betrachtet. Im ersten Gebiet sind die Richtungen von Grenzschichtströmung und Auftriebsströmung gleich, während im zweiten die Richtungen umgekehrt sind. Die vorgestellte Theorie wird durch Versuche bestätigt.

ЕСТЕСТВЕННОКОНВЕКТИВНЫЙ ТЕПЛОПЕРЕНОС У ПЛАСТИН С КОНЕЧНЫМИ РАЗМЕРАМИ

Аннотация—Предложен новый подход к описанию естественной конвекции у изотермической наклонной пластины, в рамках которого получено упрощенное аналитическое решение. В принятом описании выделяются две различные области с разными режимами течения жидкости. В первой области направление течения жидкости внутри пограничного слоя и направление параллельной пластине составляющей подъемной силы совпадают, тогда как во второй области эти направления противоположны. Предложенная теория подтверждается экспериментально.